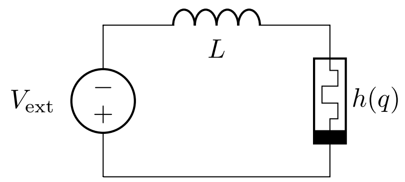
Linear systems – Final exam – Version B – Solutions

Final exam 2020–2021, Tuesday 15 June 2021, 15:00 – 18:30

Problem 1

(3 + 7 + 4 + 10 + 4 = 28 points)



Electrical circuits with nonlinear elements are expected to form the building blocks for future brain-inspired computers. The simplest such circuit, depicted above, can be modelled as

$$\begin{aligned} L\dot{I}(t) &= -\frac{\mathrm{d}h}{\mathrm{d}q}\big(q(t)\big)I(t) + V_{\mathrm{ext}}(t), \\ \dot{q}(t) &= I(t), \end{aligned} \tag{1}$$

where $I(t) \in \mathbb{R}$ is the current through the inductor with inductance $L > 0$. The nonlinear element has the internal state variable $q(t) \in \mathbb{R}$ and is characterized by the smooth function h satisfying

$$\frac{\mathrm{d}h}{\mathrm{d}q}(q) > 0 \quad \text{for all } q \in \mathbb{R}.$$

Finally, $V_{\mathrm{ext}}(t) \in \mathbb{R}$ is the external voltage applied to the circuit.

(a) The total energy in the circuit is given by the function $E(I) = \frac{1}{2}LI^2$. Let $V_{\mathrm{ext}}(t) = 0$ for all t. Show that the circuit dissipates energy, i.e., the total energy $E(I(t))$ is not increasing as a function of time.

Answer. By direct computation,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\big\{E(I(t))\big\} = LI(t)\dot{I}(t) = -\frac{\mathrm{d}h}{\mathrm{d}\varphi}\big(q(t)\big)\big(I(t)\big)^2 \leq 0, \tag{2}$$

i.e., the total energy cannot increase.

(b) Show that, for any $\bar{q} \in \mathbb{R}$, $(I, q) = (0, \bar{q})$ is an equilibrium point for the constant input $V_{\mathrm{ext}}(t) = \bar{V}_{\mathrm{ext}}$ with $\bar{V}_{\mathrm{ext}} = 0$. Moreover, linearize the dynamics (1) around the equilibrium $(0, \bar{q})$ for $\bar{V}_{\mathrm{ext}} = 0$.

Answer. Subsitution of $(I, q) = (0, \bar{q})$ in the right-hand sides of (1) gives

$$-\frac{\mathrm{d}h}{\mathrm{d}\varphi}(\bar{q}) \cdot 0 = 0, \qquad 0 = 0, \tag{3}$$

such that we have $(0, \bar{q})$ is indeed an equilibrium for any $\bar{q} \in \mathbb{R}$.

Continuing with linearization, denote

$$x(t) = \begin{bmatrix} I(t) \\ q(t) \end{bmatrix}, \quad u(t) = V_{\mathrm{ext}}(t), \qquad \bar{x} = \begin{bmatrix} 0 \\ \bar{q} \end{bmatrix}, \quad \bar{u} = \bar{V}_{\mathrm{ext}}, \tag{4}$$

and introduce the perturbations

$$\tilde{x}(t) = x(t) - \bar{x}, \quad \tilde{u}(t) = u(t) - \bar{u}. \tag{5}$$

Using the above notation, we write

$$f(x,u) = \begin{bmatrix} -C^{-1}\frac{\mathrm{d}h}{\mathrm{d}q}(q)v + L^{-1}V_{\mathrm{ext}} \\ I \end{bmatrix} \tag{6}$$

such that the dynamics (1) can be written in the standard state-space form $\dot{x} = f(x,u)$. Now, we know that the linearization is obtained as

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t), \tag{7}$$

where

$$A = \frac{\partial f}{\partial x}(\bar{x},\bar{u}), \quad B = \frac{\partial f}{\partial u}(\bar{x},\bar{u}). \tag{8}$$

To obtain expressions for A and B, note that

$$\frac{\partial f}{\partial x}(x,u) = \begin{bmatrix} -L^{-1}\frac{\mathrm{d}h}{\mathrm{d}q}(q) & -L^{-1}\frac{\mathrm{d}^2h}{\mathrm{d}q^2}(q)I \\ 1 & 0 \end{bmatrix}, \qquad \frac{\partial f}{\partial u}(x,u) = \begin{bmatrix} L^{-1} \\ 0 \end{bmatrix} \tag{9}$$

such that

$$\frac{\partial f}{\partial x}(\bar{x},\bar{u}) = \begin{bmatrix} -L^{-1}\frac{\mathrm{d}h}{\mathrm{d}q}(\bar{q}) & 0 \\ 1 & 0 \end{bmatrix}, \qquad \frac{\partial f}{\partial u}(\bar{x},\bar{u}) = \begin{bmatrix} L^{-1} \\ 0 \end{bmatrix}. \tag{10}$$

(c) Using the linearization in (b), show that the equilibrium $(0,\bar{q})$ for $\bar{V}_{\mathrm{ext}} = 0$ is not asymptotically stable.

Answer. From the zero column in (10), it is clear that $0 \in \sigma(A)$. Consequently, $\sigma(A) \not\subset \mathbb{C}_-$ and the equilibrium is not asymptotically stable.

In the remainder of this problem, we consider the initial value problem (1) with initial conditions

$$I(0) = I_0, \qquad q(0) = q_0. \tag{11}$$

and $V_{\mathrm{ext}}(t) = 0$ for all $t \geq 0$.

(d) By explicitly solving the first equation in (1), show that the initial value problem (1), (11) can be written in the simpler form

$$\dot{q}(t) = I_0 - \frac{1}{L}\big(h(q(t)) - h(q_0)\big), \qquad q(0) = q_0. \tag{12}$$

In addition, show how $I(t)$ can be obtained from a solution $q(t)$ to (12).

Answer. Note that the first equation in (1), for $V_{\mathrm{ext}}(t) = 0$, can be written as

$$L\dot{I}(t) = -\frac{\mathrm{d}h}{\mathrm{d}q}\big(q(t)\big)I(t) = -\frac{\mathrm{d}h}{\mathrm{d}q}\big(q(t)\big)\dot{q}(t) = -\tfrac{\mathrm{d}}{\mathrm{d}t}\big\{h(q(t))\big\}, \tag{13}$$

by use of the chain rule and the second equation in (1). Integration of the result over the interval $[0,t]$ gives

$$\int_0^t L\dot{I}(\tau)\,\mathrm{d}\tau = \int_0^t -\tfrac{\mathrm{d}}{\mathrm{d}\tau}\big\{h(q(\tau))\big\}\,\mathrm{d}\tau, \tag{14}$$

such that

$$L\big(I(t) - I(0)\big) = -h(q(t)) + h(q(0)). \tag{15}$$

Rearranging terms and the use of the initial conditions (11) leads to

$$I(t) = I_0 - L^{-1}\big(h(q(t)) - h(q_0)\big), \tag{16}$$

after which substitution in the second equation in (1) leads to

$$\dot{q}(t) = I_0 - L^{-1}\big(h(q(t)) - h(q_0)\big). \tag{17}$$

This is indeed the desired result. Here, we stress that (17) is still a differential equation and thus requires the initial condition $q(0) = q_0$.

Finally, note that (16) allows for constructing $I(t)$ from any solution $q(t)$ to (17).

(e) It can easily be shown that the dynamics in (12) has a *unique* equilibrium point q^* and that the solution to the initial value problem (12) satisfies

$$\lim_{t\to\infty} q(t) = q^*.$$

Why does this not contradict the answer in (c)?

Answer. The dynamics (12) holds *only* for the initial condition (11) and the equilibrium point in (12) is dependent on this initial condition. For clarity, denote this as $q^*(I_0, q_0)$. We thus have

$$\lim_{t\to\infty} q(t) = q^*(I_0, q_0). \tag{18}$$

The above gives the more useful interpretation of q^* as the limit value of $q(t)$ for given initial conditions. As this limit value is generally dependent on the initial conditions, the result is not in disagreement with (c).

Problem 2 (15 points)

Consider the linear system given by the transfer function

$$T(s) = \frac{s+2}{s^4 + as^3 + 4as^2 + 2as + 3a}$$

with $a \in \mathbb{R}$. Give the values of a for which the system is externally stable.

Answer. Denote

$$q(s) = s^4 + as^3 + 4as^2 + 2as + 3a. \tag{19}$$

First, note that external stability is determined by the roots of the denominator polynomial q *after cancelling factors common with the numerator.* However, as -2 is the only root of the numerator polynomial, it can at most cancel a stable root of q. As such, the system given by the transfer function T is stable if and only if q is a stable polynomial.

We proceed by forming the following Routh-Hurwitz table:

		s^4	s^3	s^2	s^1	s^0	
$a\times$	q	1	a	$4a$	$2a$	$3a$	(Step 0)
$1\times$		a		$2a$			
	r		a^2	$4a^2-2a$	$2a^2$	$3a^2$	(Step 1)
$(4a-2)\times$	r'		a	$4a-2$	$2a$	$3a$	(Step 1')
$a\times$			$4a-2$		$3a$		
	t			$(4a-2)^2$	b	$3a(4a-2)$	(Step 2)

From the table, we can draw the following conclusions.

Step 0. Recall that a necessary condition for stability of a polynomial is that all its coefficients have the same sign. Hence, already from q we can conclude that

$$a > 0 \tag{20}$$

is a necessary condition. We assume this from now on. Note that this also allows for application of the first step in the recursive procedure.

Step 1. The polynomial r is the result after applying the first step. To simplify this, note that $a > 0$ by assumption, allowing division by a.

Step 1'. Using a similar reasoning as in Step 0, we see that a necessary condition for stability of r' is that

$$4a - 2 > 0 \qquad \Leftrightarrow \qquad a > \tfrac{1}{2} \tag{21}$$

We proceed with one more step, as is allowed as the two leading coefficients have the same sign.

Step 2. The result of step 2 is the polynomial t, where

$$b = 2a(4a-2) - 3a^2 = 5a^2 - 4a = a(5a-4). \tag{22}$$

We know that a second-order polynomial (as t is) is stable if and only if all coefficients have the same sign. As we had already assumed (21), this is the case if and only if $a > \frac{4}{5}$ as follows from (22).

Combining the above, we have that $a > \frac{4}{5}$ is necessary and sufficient for t to be stable. Through the Routh-Hurwitz procedure, this implies stability of r and q. Hence, q is stable if and only if

$$a > \tfrac{4}{5}. \tag{23}$$

By the reasoning on top of this page, this is also necessary and sufficient for external stability of T.

Problem 3 (4 + 10 + 8 = 22 points)

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t).$$

with state $x(t) \in \mathbb{R}^2$, input $u(t) \in \mathbb{R}$, and matrices

$$A = \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(a) Verify that the system is controllable.

Answer. A direct computation shows that

$$\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} \tag{24}$$

such that

$$\operatorname{rank} \begin{bmatrix} B & AB \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} = 2 = n, \tag{25}$$

i.e., the system is controllable.

(b) Find a nonsingular matrix T and real numbers α_1, α_2 such that

$$TAT^{-1} = \begin{bmatrix} 0 & 1 \\ \alpha_1 & \alpha_2 \end{bmatrix}, \qquad TB = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hint. It is sufficient to give T^{-1}.

Answer. We first compute the characteristic polynomial of A as

$$\begin{aligned} \Delta_A(s) = \det(sI - A) &= \begin{vmatrix} s - \frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & s - \frac{1}{2} \end{vmatrix} \\ &= (s - \tfrac{3}{2})(s - \tfrac{1}{2}) + \frac{9}{4} \\ &= s^2 - 2s + \tfrac{3}{4} + \tfrac{9}{4} \\ &= s^2 - 2s + 3 \end{aligned} \tag{26}$$

This polynomial can be written in standard form as

$$\Delta_A(s) = s^2 + a_1 s + a_0 \tag{27}$$

with $a_1 = -2$ and $a_0 = 3$.

Now, consider the vectors

$$q_2 = B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad q_1 = AB + a_1 B = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \tag{28}$$

leading to the matrix T characterized through its inverse as

$$T^{-1} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}. \tag{29}$$

By construction, this choice of matrix leads to TAT^{-1} and TB of the form (26), with

$$\alpha_1 = -a_0 = -3, \qquad \alpha_2 = -a_1 = 2. \tag{30}$$

As can be verified by direct computation. For reference, T is given as

$$T = \tfrac{1}{4} \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}. \tag{31}$$

(c) Use the matrix T from (b) to design a state feedback controller $u(t) = Fx(t)$ such that the closed-loop system matrix $A + BF$ has eigenvalues -1 and -2.

Answer. Note that

$$\Delta_{A+BF}(s) = \Delta_{T(A+BF)T^{-1}}(s), \tag{32}$$

which allows controller design in the "transformed coordinates". Furthermore, denote

$$FT^{-1} = \begin{bmatrix} f_0 & f_1 \end{bmatrix} \tag{33}$$

such that

$$\begin{aligned} T(A+BF)T^{-1} &= TAT^{-1} + TBF^{-1} \\ &= \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} f_0 & f_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ f_0 - a_0 & f_1 - a_1 \end{bmatrix}. \end{aligned} \tag{34}$$

As the above matrix is in so-called companion form, it is immediate that

$$\Delta_{T(A+BF)T^{-1}}(s) = s^2 + (a_1 - f_1)s + (a_0 - f_0). \tag{35}$$

Recall that we would like the closed-loop system matrix $A + BF$ to have eigenvalues at -1 and -2. To enforce this, define the desired polynomial p such that it has its roots at those locations. Specifically,

$$p(s) = (s+1)(s+2) = s^2 + 3s + 2. \tag{36}$$

Comparing this with (35) leads to

$$\begin{matrix} a_1 - f_1 = 3 \\ a_0 - f_0 = 2 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} f_1 = a_1 - 3 = -2 - 3 = -5 \\ f_0 = a_0 - 2 = 3 - 2 \quad = 1 \end{matrix} \tag{37}$$

such that

$$FT^{-1} = \begin{bmatrix} f_0 & f_1 \end{bmatrix} = \begin{bmatrix} 1 & -5 \end{bmatrix}. \tag{38}$$

By solving this linear equation for F (e.g., by post-muliplying by T), we obtain

$$F = \begin{bmatrix} -\frac{7}{2} & \frac{3}{2} \end{bmatrix}. \tag{39}$$

Problem 4 (5 + 5 = 10 points)

Consider the linear system

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ -4 & 2 & 1 \end{bmatrix} x(t), \qquad y(t) = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix} x(t).$$

(a) Show that the system is not observable and give a basis for the unobservable subspace.

Answer. Compute

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 1 \\ -9 & 2 & -1 \\ 15 & -2 & 1 \end{bmatrix}. \tag{40}$$

As the third column is clearly linearly dependent from the second column, it is clear that this matrix does not have full column rank. As such, it is not observable.

Recall that the unobservable subspace $\mathcal{N}$ is given as

$$\mathcal{N} = \ker \begin{bmatrix} 3 & -2 & 1 \\ -9 & 2 & -1 \\ 15 & -2 & 1 \end{bmatrix}, \tag{41}$$

for which is a basis is easily obtained as

$$\mathcal{N} = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}. \tag{42}$$

(b) Is the system detectable?

Answer. As a first step in determining detectability, we need to compute the eigenvalues of A. Due to the lower block triangular structure, we have

$$\sigma(A) = \sigma\left(\begin{bmatrix} -1 \end{bmatrix}\right) \cup \sigma\left(\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\right) = \{-1\} \cup \sigma\left(\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\right). \tag{43}$$

To compute the spectrum of the 2×2 matrix above, consider its characteristic polynomial

$$\begin{vmatrix} s & -1 \\ -2 & s-1 \end{vmatrix} = s(s-1) - 2 = s^2 - s - 2 = (s-2)(s+1). \tag{44}$$

As a result, we have

$$\sigma\left(\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\right) = \{-1, 2\}, \tag{45}$$

such that

$$\sigma(A) = \{-1, 2\} \tag{46}$$

where -1 has multiplicity two. It is clear that $\lambda = 2$ is the only eigenvalue for which $\mathrm{Re}(\lambda) > 0$. Following the Hautus test for detectability, compute

$$\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 1 & -2 & 1 \\ -4 & 2 & -1 \\ 3 & -2 & 1 \end{bmatrix} \tag{47}$$

from which it can be concluded that

$$\operatorname{rank}\begin{bmatrix} A-\lambda I \\ C \end{bmatrix} = \operatorname{rank}\begin{bmatrix} -3 & 0 & 0 \\ 1 & -2 & 1 \\ -4 & 2 & -1 \\ 3 & -2 & 1 \end{bmatrix} = 2 < 3 = n, \tag{48}$$

such that the system is not detectable.

Problem 5 (15 points)

Consider the linear system

$$\dot{x}(t) = Ax(t), \qquad y(t) = Cx(t) \tag{49}$$

with $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$. The linear system is called *output stable* if

$$\lim_{t\to\infty} Ce^{At}x_0 = 0 \quad \text{for all } x_0 \in \mathbb{R}^n.$$

Show that the following two statements are equivalent:

1. (49) is output stable;
2. every eigenvalue λ of A satisfying $\mathrm{Re}(\lambda) \geq 0$ is not (A, C)-observable.

Hint. You may use the following fact: for an eigenvector v corresponding to an eigenvalue λ of A, we have that $e^{At}v = ve^{\lambda t}$.

Answer. We start with the proof of 1. ⇒ 2.. To prove 2., let λ be an eigenvalue of A satisfying $\mathrm{Re}(\lambda) \geq 0$ and let v be an associated eigenvalue, i.e., $Av = \lambda v$ Choosing x_0 and using the hint, we get from output stability (in 1.) that

$$0 = \lim_{t\to\infty} Ce^{At}v = \lim_{t\to\infty} Cve^{\lambda t}. \tag{50}$$

However, as $\mathrm{Re}(\lambda) \geq 0$, this implies $Cv = 0$. Collecting what we have, we can write

$$\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} v = 0, \tag{51}$$

which implies

$$\mathrm{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} < n \tag{52}$$

as $v \neq 0$ by definition of an eigenvalue. Hence, λ is not (A, C)-observable.

The implication 2. ⇒ 1. is not true. A counterexample is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{53}$$

The only eigenvalue $\lambda = 1$ is not (A, C)-observable, but the corresponding system is not output stable.

(10 points free)